Project 10: Equality and Equivalence, and Solutions to Equations

A **solution** to an equation (or inequality, or system of equations) is the **set of all values of the variable(s) that makes the equation <u>true</u>**. So, for example:

Example1:
The solution to the equation $2x + 1 = 9$ is $x = 4$, since the equation is **only** true when $x = 4$, and is NOT true for any other values of x.

Example 2:
The equation $x = x + 5$ has **no solution**, because there is **NO** value of x that will make the equation true. (Because we can see that no number will be equal to itself plus five.)

Example 3:
The equation $2x = x + x$ has an **infinite number of solutions**—x can be **any** real number and the equation will always be true, no matter which value we choose for x. (Because we can see that anything multiplied by two will be the same as adding two copies of it, by the very definition of multiplication.)

Example 4:
For the equation $x^2 + 6 = -5x$, the solution is $x = -2, x = -3$ because the equation is **only** true when $x = -2$ or $x = -3$, and is NOT true for any other values of x.

Example 5:
For the inequality $2(x - 5) < 3x + 6$, the solution is $x > 16$ because the inequality is **only** true for values of x that are larger than 16, and it is NOT true when x is 16 or less than 16. (So, for example, 20, 58.2604739, and 1,000,001 are all in the solution set for this inequality, since they are all greater than 16).

Example 6:
For the system of equations $x - y = 2, -2x + 3y = 4$, the solution to the system of equations is $x = 10, y = 8$ or $(10,8)$, since this is the **only** combination of x and y that make **both** equations true. For any other combination of x and y values, at least one of the equations will be false. This is also the point on the graph where the two lines represented by each equation would intersect.

Example 7:
For the system of equations $x - y = 3, x - y = 5$ there is **no solution**, because there are **NO** values for x and y that can make both of these equations true at the same time (because we can see that if the difference between two numbers is 3, then it cannot also be 5 at the same time). In this case, the two lines represented by each of these equations never intersect—they are parallel to each other.

Example 8:
For the system of equations $x + y = 3, x - 3 = y$ there are **infinitely many solutions**—for every possible real number that we plug in for x, we can find a y-value that will make both equations true, and for any real number that we plug in for y, we can find an x-value that will make both equations true (because we can see that if two numbers add together to get 3, subtracting three from one of the numbers will always give us the other number). The two lines represented by these two equations are actually the same line, because the two equations are equivalent, so all points on one line are also on the other line.

So, if we have a solution to an equation (or an inequality, or a system of equations) we can always check that it is really a solution by substituting it into the equation (or the inequality, or both equations in the system) to check that it makes the equation (or inequality, or both equations in the equality) true. Let’s try this with several examples:

Now you try!

Which of the following are solutions to the given equation, inequality, or system of equations?

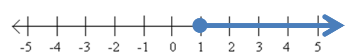
1. Consider the equation $y^2 - 12 = -x$. Which of the following gives **ALL the solutions** to this equation? Show your work or give an explanation for each option to explain why it is or is not the solution set:
   a. $x = 3, x = -4$

   b. $x = -3, x = 4$

   c. $x = -3, x = -4$

   d. $x = 3, x = 4$

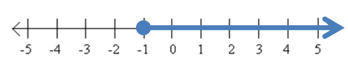
2. Which of the following graphs cannot possibly represent the solution to the inequality? One of them is correct—find out which by choosing some values in the solution set for each possible graph to show which of these can NOT possibly be the solution, by testing those values in the inequality: $-2x + 6 \leq 7 - x$. Then cross out those graphs that can NOT be the solution, and circle the one remaining that is the only remaining possible solution. Show your work or give an explanation for each option to explain why it is or is not the solution set:

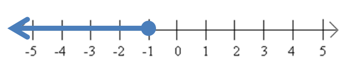
   a.

   b.

   c.

   d.

3. Consider the equation $2x^3 + 12 = 56$. Which of the following is a solution for x that makes the equation true? Test each possible solution by substituting it into the equation. Show your work or give an explanation for each option to explain why it is or is not the solution set:
   a. $x = 12$

   b. $x = 32$

   c. $x = 23$

   d. $x = 2$

4. Consider the equation $3 - 9x = -7(x + 5)$. Which of the following is a solution for x that makes the equation true? Test each possible solution by substituting it into the equation. Show your work or give an explanation for each option to explain why it is or is not the solution set:
   a. $x = 19$

   b. $x = -16$

   c. $x = 16$

   d. $x = -1$

5. Consider the equation $z = 2x - 5y$. Which of the following is a solution for x that makes the equation true? Test each possible solution by substituting it into the equation. Show your work or give an explanation for each option to explain why it is or is not the solution set:
   a. $x = 2(z + 5y)$

   b. $x = z^2 + 5y$

   c. $x = \frac{z-5}{y^2}$

   d. $x = \frac{z+5}{y^2}$

6. Consider each of the equations below. Which of these equations represents a line that contains both the points $(1, -2)$ and $(-2, 7)$? Test out each equation by substituting in first one point, and then (if the first point makes the equation true) try the second point after that. Show your work or give an explanation for each option to explain why it is or is not the solution set:
   a. $y = 3x + 13$

   b. $y = 3x - 5$

   c. $y = -3x - 2$

   d. $y = -3x + 1$

7. Consider the system of equations $\begin{array}{c} -2x + y = 2 \\ 8x - 5y = 4 \end{array}$, and the following values for x. One of these x-values is a solution to this system—identify which one! In order to find out which one, substitute each x-value into one of the equations and solve for y. Then substiutute that set of x and y values into the other equation to test whether that pair of x and y values works for both equations. <u>Show your work or give an explanation for each option</u> to explain why it is or is not the solution set:

   a. $x = 7$

   b. $x = -3$

   c. $x = 3$

   d. $x = -7$

Factoring

Factoring is just reversing multiplication. So for example, if $(2x-1)(x-7)=2x^2-15x+7$, then factoring $2x^2-15x+7$ just means rewriting it as $(2x-1)(x-7)$. Sometimes this is easier, and sometimes it is trickier to find the factors. Often, we can just find the factors by applying the distributive property in reverse, as we have already done, by using the identity $ab+ac=a(b+c)$ or $ab+ac+ad=a(b+c+d)$. (Or their similar properties $ac+bc=(a+b)c$ and $ac+bc+dc=(a+b+d)c$.)

Example 1: $15x^4-35x^3+10x^2$ can be factored this way:
$= 3\cdot 5\cdot x\cdot x\cdot x\cdot x + -1\cdot 7\cdot 5\cdot x\cdot x\cdot x + 2\cdot 5\cdot x\cdot x\cdot x$
$= \boxed{5\cdot x\cdot x}\cdot 3\cdot x\cdot x + \boxed{5\cdot x\cdot x}\cdot -1\cdot 7\cdot x + \boxed{5\cdot x\cdot x}\cdot 2$
$= (5x^2)(3x^2)+(5x^2)(-7x)+(5x^2)(2)$
$= (5x^2)(3x^2+-7x+2)$
We might be able to factor this second part further, if we can only figure out what the factors are that need to be multiplied to get this expression $3x^2+-7x+2$. We will return to this in a minute.

Example 2: $8ax-4ay+12bxw-6by$ can be factored the following way:
$8ax-4ay+12bx-6by$
$= 8ax+-4ay+12bx+-6by$
First we look at the first two terms as a group and the last two terms as a group, and we look for common factors in each pair:
$= \boxed{(4a)(2x)+(4a)(-y)} + \underline{(6b)(2x)+(6b)(-y)}$
$= \boxed{(4a)(2x+-y)} + \underline{(6b)(2x+-y)}$
$= (4a)(2x+-y)+(6b)(2x+-y)$
Now we notice that the second factor, $(2x+-y)$ is the same in each half, so we can apply the distribute property in reverse again this way:
$= (4a+6b)(2x+-y)$

Sometimes we can't see what has been multiplied together when we are trying to factor. For example, think about this example:
If we were to multiply $(3x-1)(x-2)$, we would get:
$= (3x+-1)(x+-2)$
$= (3x+-1)(x)+(3x+-1)(-2)$
$= (3x)(x)+(-1)(x)+(3x)(-2)+(-1)(-2)$
$= 3x^2+-1x+-6x+2$
$= 3x^2+(-1+-6)x+2$
$= 3x^2+-7x+2$

So, if we were told to factor $3x^2+-7x+2$ we could just replace it with $(3x-1)(x-2)$! But if we didn't originally do the multiplying, how can we figure out that this is what was multiplied?

There are some general approaches to do this, which you can see in project 11. But for now, let's stick to a simple example where we are given some possible options, where we could try each one:

Example 3:

Which of the following is a factor of $3x^2 + -7x + 2$?

a. $x + 2$
b. $3x + 2$
c. $3x - 2$
d. $x - 1$

The easiest way to figure out which one will work is to come up with a second factor pair that will make the multiplication work out for the first and last terms of the expression that we are trying to factor, and then to multiply things out and see if we get what we started with. So, for example:

a. $x + 2$

This one starts with x, and if we are going to multiply x by something to get $3x^2$, it would have to be $3x$ (because $x \cdot 3x = 3x^2$). The second term here is 2, and if we are going to multiply 2 by something to get 2, it would have to also be 1 (because $2 \cdot 1 = 2$). So, $x + 2$ can only be a factor if $(x + 2)(3x + 1) = 3x^2 + -7x + 2$. But we noticed that multiplying $(x + 2)(3x + 1)$ is actually:

$$(x + 2)(3x + 1) = (x)(3x + 1) + (2)(3x + 1) = (x)(3x) + (x)(1) + (2)(3x) + (2)(1)$$
$$= 3x^2 + 1x + 6x + 2 = 3x^2 + (1 + 6)x + 2 = 3x^2 + 7x + 2$$

In fact, before we even multiply all this out, we can see that $(x + 2)(3x + 1) \neq 3x^2 + -7x + 2$ because of the $-7x$ term: because ALL of the terms in the first two factors are positive, there is no way we can multiply these to get a negative term. So we know that $x + 2$ is NOT a factor.

b. $3x + 1$

Similarly to a. above, we can see that $3x + 1$ can only be a factor if $(3x + 1)(x + 2) = 3x^2 + -7x + 2$. But we just showed in a. that this is not true, so we know that $3x + 1$ is NOT a factor.

c. $3x - 1$

Similarly to a. above, we can see that $3x - 1$ can only be a factor if $(3x + -1)(x + -2) = 3x^2 + -7x + 2$. Let's test that out:

$$(3x + -1)(x + -2) = (3x)(x + -2) + (-1)(x + -2) = (3x)(x) + (3x)(-2) + (-1)(x) + (-1)(-2)$$
$$= 3x^2 + -6x + -1x + 2 = 3x^2 + (-6 + -1)x + 2 = 3x^2 + -7x + 2 = 3x^2 - 7x + 2$$

So, $3x - 1$ IS a factor!

d. $x - 1$

Similarly to a. above, we can see that $x - 1$ can only be a factor if $(x + -1)(3x + -2) = 3x^2 + -7x + 2$. Let's test that out:

$$(x + -1)(3x + -2) = (x + -1)(3x) + (x + -1)(-2) = (x)(3x) + (-1)(3x) + (x)(-2) + (-1)(-2)$$
$$= 3x^2 + -3x + -2x + 2 = 3x^2 + (-3 + -2)x + 2 = 3x^2 + -5x + 2$$

This isn't what we need to get when we multiply, so $x - 1$ is NOT a factor!

Example 4:

Let's put examples 1 and 3 together now:

Factor $15x^4 - 35x^3 + 10x^2$ completely:

a. $(5x^2)(3x^2 + -7x + 2)$
b. $(x^2)(15x - 5)(x - 2)$
c. $(5x^2)(3x + 1)(x + 2)$
d. $(5x^2)(3x - 1)(x - 2)$

To do this problem, we will multiply out each of these:

a. $(5x^2)(3x^2 + -7x + 2) = (5x^2)(3x^2) + (5x^2)(-7x) + (5x^2)(2) = 5 \cdot 3 \cdot x \cdot x \cdot x \cdot x + 5 \cdot -7 \cdot x \cdot x \cdot x + 5 \cdot 2 \cdot x \cdot x = 15x^4 + -35x^3 + 10x^2 = 15x^4 - 35x^3 + 10x^2$

So this is a correct factoring. But is it factored completely? In other words, has this broken it down into the smallest factors? Let's look at the other options to see if there are any other correct factorings here…

b. $(x^2)(15x - 5)(x - 2)$
First I notice that I can rewrite $(15x - 5) = (15x + -5) = (3x \cdot 5 + -1 \cdot 5) = (3x + -1)(5)$.
So we have $(x^2)(15x - 5)(x - 2) = (x^2)(5)(3x + -1)(x + -2) = (5x^2)(3x + -1)(x + -2)$
This is just the same as option d. below. If we multiply out $(3x + -1)(x + -2)$, we get:
$(3x + -1)(x + -2) = 3x^2 + -7x + 2$, which is just the same as what we have in a. above:
$(x^2)(15x - 5)(x - 2) = (5x^2)(3x + -1)(x + -2) = (5x^2)(3x^2 + -7x + 2)$.
So it looks like a., b., and d. are all factorizations of $15x^4 - 35x^3 + 10x^2$. Which of these three has been factored the most? Option d. has, because it has the SMALLEST factors (with the smallest numbers and the fewest exponents). So we need to check option c. first, but so far option d. looks like our best bet.

c. $(5x^2)(3x + 1)(x + 2) = (5x^2)[(3x + 1)(x) + (3x + 1)(2)] = (5x^2)[(3x)(x) + (1)(x) + (3x)(2) + (1)(2)] = (5x^2)[3x^2 + 1x + 6x + 2] = (5x^2)[3x^2 + (1 + 6)x + 2] = (5x^2)[3x^2 + 7x + 2] = (5x^2)[3x^2] + (5x^2)[7x] + (5x^2)[2] = 5 \cdot 3 \cdot x \cdot x \cdot x \cdot x + 5 \cdot 7 \cdot x \cdot x \cdot x + 5 \cdot 2 \cdot x \cdot x = 15x^4 + 35x^3 + 10x^2$
But $15x^4 + 35x^3 + 10x^2 \neq 15x^4 - 35x^3 + 10x^2$, so this can NOT be a factoring of $15x^4 - 35x^3 + 10x^2$!
We could actually see this before multiplying out everything: because all of the terms in all of the factors here are positive, there is no way for us to produce the $-35x^3$ in the original equation that we were trying to factor.

d. $\boxed{(5x^2)(3x - 1)(x - 2)}$ This is the complete factoring of the original expression, as we say in a. and b.!

Now you try!

For each of the following problems, 1) **multiply** the factors out to determine which are factors of the original expression. 2) If there is more than one set of factors, **circle the one** that has been **most completely factored**.
If only one possible factor is given, **first generate the other possible factor** that would need to go with it to generate the correct first and last term, and then **test that pair of factors by multiplying them out**.

8. Multiply. $(4x - 2)(x^2 - 5x + 3)$.
   a. $4x^3 - 22x^2 + 22x - 6$

   b. $4x^3 - 20x^2 + 22x - 6$

   c. $4x^3 - 22x^2 + 12x - 6$

   d. $4x^3 - 20x^2 + 12x - 6$

9. Factor completely. $8x^2y - 18y^3$ Show your work or give an explanation for each option to explain why it is or is not the complete factorization:
   a. $2(4x^2y - 9y^3)$

   b. $2y(4x - 9y)(4x + 9y)$

   c. $2y(2x - 3y)(2x - 3y)$

   d. $2y(2x - 3y)(2x + 3y)$

10. Which of the following is a factor of the polynomial? $5x^2 + 13x - 6$ Show your work or give an explanation for each option to explain why it is or is not a factor:
    a. $x - 3$

    b. $5x + 2$

    c. $5x - 3$

    d. $x + 3$

11. Which of the following is a factor of the polynomial? $15cw - 20cz - 6dw + 8dz$ Show your work or give an explanation for each option to explain why it is or is not a factor:
    a. $5c + 2d$

    b. $5c + 4d$

    c. $3w - 4z$

    d. $3w + 4z$